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Semigroup rings and simplicial complexes

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Dedicated to the memory of Professor Hideyuki Matsumura

Abstract

We study the minimal free resolution $\mathscr{F}$ of a ring $T = S/I$ where S is a positive affine semigroup ring over a field K, and I is an ideal in S generated by monomials. We will essentially use the fact that the multigraded Betti numbers of T can be computed from the relative homology of simplicial complexes that we shall call *squarefree divisor complexes*. In a sense, these simplicial complexes represent the divisibility relations in S if one neglects the multiplicities with which the irreducible elements appear in the representation of an element. In Section 1 we study the dependence of the free resolution on the characteristic of K. In Section 2 we show that, up to an equivalence in homotopy, every simplicial complex can be 'realized' in a normal semigroup ring and also in a one-dimensional semigroup ring. Furthermore, we describe all the graphs among the squarefree divisor complexes. In Section 3 we deduce assertions about certain simplicial complexes of chessboard type from information about free resolutions of well-understood semigroup rings. © 1997 Elsevier Science B.V.

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In this note we study the minimal free resolution $\mathscr{F}$ of a ring $T = S/I$ where S is a positive affine semigroup ring over a field K, and I is an ideal in S generated by monomials. We will essentially use the fact that the multigraded Betti numbers of T can be computed from the relative homology of simplicial complexes that we shall call *squarefree divisor complexes*. In a sense, these simplicial complexes represent the divisibility relations in S if one neglects the multiplicities with which the irreducible elements appear in the representation of an element.

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The connection between the Betti numbers of multigraded algebras and Betti numbers of simplicial complexes is not new; e.g., it has been applied by Hochster [10], Campillo and Marijuan [5], and Anderson [1]. In Section 1 we will use it to study the dependence of the free resolution on the characteristic of K.

In Section 2 we show that, up to an equivalence in homotopy, every simplicial complex can be 'realized' in a normal semigroup ring and also in a one-dimensional semigroup ring. This result indicates that the divisibility theory of affine, even normal, semigroups is arbitrarily complex: up to homotopy, every simplicial complex arises from the decompositions of their elements into irreducible elements.

While the exact classification of the simplicial complexes arising as squarefree divisor complexes is probably very difficult, we succeed in describing all the graphs among them, and also the significantly smaller class of those graphs that appear in normal semigroups.

In Section 3 we deduce assertions about certain simplicial complexes of chessboard type from information about free resolutions of well-understood semigroup rings.

1. Betti numbers and characteristic

Let K be a field. A subalgebra of the polynomial ring $K[Y_1,\dots,Y_m]$ over the field K generated by a finite number of monomials $y_1,\dots,y_n$ is called a *positive affine semigroup ring*. In the following S will always denote such a ring. The monomials contained in S form a semigroup under multiplication, and the function $\deg : \mathcal{M} \to \mathbb{N}^m$ that assigns each monomial its exponent vector maps $\mathcal{M}$ isomorphically onto a subsemigroup H of $\mathbb{N}^m$. Up to isomorphism, S is the semigroup algebra $K[H]$. We will always assume that $y_1,\dots,y_n$ are irreducible elements of $\mathcal{M}$, or, in other words, that they form a minimal system of algebra generators of S.

Let Σ be the simplex with vertex set $\{1,\dots,n\}$. For a face F of Σ we let y^F denote the product of the y_i with $i \in F$, and, given an element $h \in H$, we define the *squarefree divisor complex of Y^h* to be the simplicial complex

$$\Delta_h = \{F \in \Sigma:\ y^F \text{ divides } Y^h\}.$$

Let I be an ideal of S generated by monomials $\mu \in \mathcal{M}$, and set $T = S/I$. The free resolution $\mathcal{F}$ that we will investigate is taken with respect to a representation $T \cong R/J$ where $R = K[X_1,\dots,X_n]$ is a polynomial ring whose indeterminates are mapped to the elements $y_1,\dots,y_n$. A notable special case is that in which $S = K[X_1,\dots,X_n] = K[Y_1,\dots,Y_m]$, and T is just the residue class ring of a polynomial ring modulo an ideal generated by indeterminates.

By assigning the degree of y_i to X_i we make R a multigraded K-algebra with grading group $\mathbb{Z}^m$, for which T is a multigraded R-module. Therefore $\mathcal{F}$ has a multigraded structure; its ith free module F_i decomposes into a direct sum $\bigoplus_{h\in H} R(-h)^{\beta_{ih}}$; hence $\operatorname{Tor}_i^r(K,T) \cong \bigoplus_{h\in H} K(-h)^{\beta_{ih}}$. On the other hand, the minimal free resolution of K over R is given by the Koszul complex $\mathcal{K}(X,R)$, and since Tor can be computed from a free resolution of either of its arguments, the *graded Betti number* β_{ih} can be determined from the Koszul homology; in fact,

$$H_i(\mathscr{K}(X,T)_h) \cong \bigoplus_{h\in H} K(-h)^{\beta_{ih}}$$

as well. (The index $h\in H$ denotes the degree h multigraded component.) In the special situation under consideration where $\dim_K T \le 1$ for all $h\in H$, the multigraded components of $\mathscr{K}_i(X,T)$ have a purely combinatorial description. In the following, $\tilde{\mathscr{C}}$ denotes the oriented augmented chain complex, and $\tilde{H}$ the (relative) simplicial homology or cohomology.

Proposition 1.1. *For $h\in H$ we set $\Gamma_h = \{F\in\Delta_h\colon Y^h/y^F \in I\}$. Then the following hold:*
(1) $\mathscr{K}(X,T)_h \cong (\tilde{\mathscr{C}}(\Delta_h,K)/\tilde{\mathscr{C}}(\Gamma_h,K))(-1)$,
(2) $\beta_{ih} = \dim_K \tilde{H}_{i-1}(\Delta_h,\Gamma_h,K)$.

Proof. The ith free module in the Koszul complex $\mathscr{K}(X)$ has the multigraded decomposition

$$\mathscr{K}_i(X) = \bigoplus_{F\in\Sigma,\ |F|=i} R(-\deg y^F),$$

and the differentiation $\mathscr{K}_i(X)\to\mathscr{K}_{i-1}(X)$ is given on the component $R(-\deg y^F)\to R(-\deg y^{F'})$ as the multiplication by $\varepsilon(F,F')y_j$ where $\varepsilon(F,F')=0$ if $F'\not\subset F$ and $\varepsilon(F,F')=(-1)^{k-1}$ if $F'=F\setminus\{j_k\}$, $F=\{j_1,\dots,j_i\}$, $j_1<\cdots<j_i$.

We obtain $\mathscr{K}_i(X,S)$ and $\mathscr{K}_i(X,I)$ by replacing R by S and by I. Let us fix a degree $h\in H$. In order to have $S(-\deg y^F)_h\neq 0$ we must have $h-\deg y^F\in H$, and this is equivalent to saying that $y^F|Y^h$. If so, then $S(-\deg y^F)_h$ is a one-dimensional vector space spanned by Y^h/y^F:

$$\mathscr{K}_i(X,S)_h \cong \bigoplus_{F\in\Delta_h,\ |F|=i} K\cdot Y^h/y^F.$$

With respect to the K-bases thus specified, the maps in $\mathscr{K}(X,S)_h$ are the same as those in $\tilde{\mathscr{C}}(\Delta_h,K)(-1)$. In fact $\tilde{\mathscr{C}}(\Delta_h,K)$ is the complex of vector spaces generated by the basis elements e_F, $F\in\Sigma$, such that

$$\tilde{\mathscr{C}}_{i-1}(\Delta_h,K) = \bigoplus_{F\in\Delta_h,\ |F|=i} Ke_F$$

with differentiation on the component $Ke_F\to Ke_{F'}$ given by the assignment $e_F\mapsto\varepsilon(F,F')e_{F'}$.

Similar arguments apply to $\mathscr{K}(X,I)$, and the exact sequence

$$0\to\mathscr{K}(X,I)\to\mathscr{K}(X,S)\to\mathscr{K}(X,T)\to 0$$

then yields the isomorphism (1). Eq. (2) is an immediate consequence of (1). $\square$

Let $\Delta \subset \Sigma$ be an arbitrary simplicial complex. Then we define the *dual complex* of Δ by

$$\bar{\Delta} = \{G \in \Sigma : \bar{G} \notin \Delta\};$$

here $\bar{G}$ denotes the set-theoretic complement of G with respect to the full vertex set $\{1, \ldots, n\}$.

Lemma 1.2. *Let $\Gamma \subset \Delta \subset \Sigma$ be simplicial complexes. Then*

$$\tilde{H}_i(\Delta, \Gamma, K) \cong \tilde{H}^{n-2-i}(\bar{\Gamma}, \bar{\Delta}, K) \cong \tilde{H}_{n-2-i}(\bar{\Gamma}, \bar{\Delta}, K).$$

Proof. Let $e_1, \ldots, e_n$ be a basis of the free $\mathbb{Z}$-module $L = \mathbb{Z}^n$. The exterior products $e_F = \bigwedge_{j \in F} e_j$, $F \in \Sigma$, $|F| = j$ are a basis of $\bigwedge^j L$. The multiplication in $\wedge L$ and the 'orientation map' $\bigwedge^n L \to \mathbb{Z}$, $e_1 \wedge \cdots \wedge e_n \mapsto 1$, induce an isomorphism $\bigwedge^j L \to (\bigwedge^{n-j} L)^*$ that maps e_F to $\sigma(F, \bar{F})(e_{\bar{F}})^*$. (Here * denotes the dual module and the dual basis respectively, and $\sigma(F, \bar{F})$ is defined by the equation $e_F \wedge e_{\bar{F}} = \sigma(F, \bar{F}) e_1 \wedge \cdots \wedge e_n$.) This construction (also see [3, 1.6.10]) yields the first of our isomorphisms (for arbitrary coefficients), whereas the second holds because we are taking coefficients in a field. □

Using the previous lemma, we can easily derive the following theorem on the independence of the β_{ih} from the characteristic of K.

Theorem 1.3. *With the notation introduced, the multigraded Betti numbers β_{ih} are independent of K for*

(a) $i = 0, 1, n-1, n$,

(b) $i = 2$ *if* $S \cong K[X_1, \ldots, X_n]$, *and*

(c) $i = n-2$ *if* $I = 0$.

Proof. The assertion is obvious for $i = 0$ and $i = n$. In fact, $\beta_{0h} = 1$ for $h = 0$, $\beta_{0h} = 0$ for $h \neq 0$, and β_{nh} is the dimension of a multigraded component of the socle of T. The socle is an ideal generated by the residue classes of all those y^g for which $g + \deg y_i \in \mathscr{I}$ where $\mathscr{I}$ is the semigroup ideal generated by the exponent vectors of the monomials in I. Therefore the multigraded structure of the socle is independent of K.

It is a well-known topological fact (and an easy exercise in linear algebra) that $\dim_K \tilde{H}_0(\Delta, \Gamma, K)$ is independent of K for all simplicial complexes $\Gamma \subset \Delta$. This implies the assertion for $i = 1$, and it also yields the case in which $i = n-1$ since, by the previous lemma, $\tilde{H}_{n-2}(\Delta, \Gamma, K) \cong \tilde{H}_0(\bar{\Gamma}, \bar{\Delta}, K)$.

In the situation of (b) one observes that Δ_h is the simplex on the *support* of h, i.e. the set $\{i : h_i \neq 0\}$. Therefore $\tilde{\mathscr{C}}(\Delta_h, K)$ is acyclic, and from the long exact homology sequence we get

$$\dim_K \tilde{H}_1(\Delta_h, \Gamma_h, K) = \dim_K \tilde{H}_0(\Gamma_h, K).$$

When $I=0$ we have $\Gamma_h=\emptyset$ for all $h\in H$. Therefore

$$\dim_K \tilde{H}_{n-3}(\Delta_h,\Gamma_h,K)=\dim_K \tilde{H}_1(\Sigma,\bar{\Delta}_h,K)=\dim_K \tilde{H}_0(\bar{\Delta}_h,K). \qquad \square$$

Part (b) of the theorem was proved by Bruns and Herzog [4] by a more constructive method, namely via a description of the third syzygy module of R/I. A similar argument as above was given by Hibi and Terai [12]. Pardue [11] also discusses the question as to what extent the Betti numbers are independent of K in the situation of (b). We will see below that the theorem cannot be extended to other Betti numbers.

Corollary 1.4. *All the multigraded Betti numbers of T are independent of K if*
(a) $n\le 4$, *or*
(b) $n=5$ *and* (i) $S=R$ *or* (ii) $I=0$.

Proof. It suffices to note that $n-1$ of the β_{ih} for a given h determine the last one. In fact, the alternating sum of the β_{ih} is the Euler characteristic of $\tilde{\mathscr{C}}(\Delta_h,K)/\tilde{\mathscr{C}}(\Gamma_h,K)$. $\square$

Let Π be a simplicial complex on $\{1,\dots,n\}$ with Stanley–Reisner ring $R/I(\Pi)$, $R=K[X_1,\dots,X_n]$; in this case $m=n$ and $H=\mathbb{N}^n$. The ideal $I(\Pi)$ is generated by all the monomials X^h with $\operatorname{supp} h\notin\Pi$. It has been observed in the proof of Theorem 1.3 that Δ_h is the simplex on the support of h. Furthermore Γ_h consists of all those $F\in\Delta_h$ for which $\operatorname{supp}(h-\imath_F)\notin\Pi$ where $\imath_F=\deg X^F$ denotes the *indicator* of F.

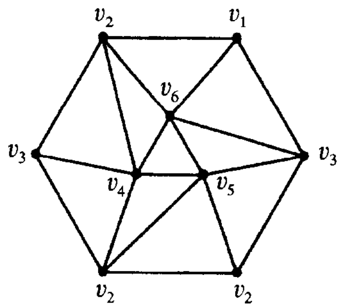
Suppose first that h is not squarefree. (We say that $h\in\mathbb{N}^n$ is *squarefree* if all its entries h_j are 0 or 1.) We pick j such that $h_j\ge 2$, and let h' be any element of $\mathbb{N}^n$ such that $h'_i=h_i$ for all $i\ne j$ and $h'_i\ge h_j$. Then $\operatorname{supp} h=\operatorname{supp} h'$ and $\operatorname{supp}(h-g)=\operatorname{supp}(h'-g)$ for every squarefree g. Thus $\Delta_h=\Delta_{h'}$ and $\Gamma_h=\Gamma'_h$. It follows that the pair (Δ_h,Γ_h) appears in infinitely many multigraded components of $\mathscr{K}(X,R/I(\Pi))$. Since only finitely many Betti numbers are non-zero, we see that $\tilde{\mathscr{C}}(\Delta_h,K)/\tilde{\mathscr{C}}(\Gamma_h,K)$ is acyclic. (There are of course several other arguments showing that only squarefree shifts occur in the minimal free resolution of $R/I(\Pi)$; for example, the (not necessarily minimal) Taylor resolution has such shifts; see [4] for a more general result on squarefree shifts.)

Next let h be a squarefree. Set $W=\operatorname{supp} h$, and let $\Pi_W=\{F\in\Pi\colon F\subset W\}$ be the restriction of Π to the vertex set W. We have $F\in\Gamma_h$ if and only if $W\setminus F\notin\Pi_W$, so that $\tilde{\Gamma}_h=\Pi_W$ where tilde denotes the dual complex with respect to the simplex Δ_h on the vertex set W. In view of the duality of Lemma 1.2 we then obtain

Corollary 1.5 (Hochster [10]). *If the multigraded Betti number β_{ih} of the Stanley–Reisner ring $R/I(\Pi)$ of the simplicial complex Π is non-zero, then h is squarefree, and*

$$\beta_{ih}=\dim_K \tilde{H}_{|W|-i-1}(\Pi_W,K),\quad W=\operatorname{supp} h.$$

Example 1.6. It has been noticed several times in the literature that the minimal triangulation Π



of the real projective plane $\mathbb{RP}^2$ is a counter-example to the independence of the Betti numbers (and the Cohen–Macaulay property) from K. Its Stanley–Reisner ring is the residue class ring of $K[X_1,\dots,X_6]$ modulo the ideal $I(\Pi)$ generated by the 10 monomials of degree 3 representing the minimal non-faces of Π. Since

$$\dim_K \tilde{H}_1(\Pi,K) = \dim_K \tilde{H}_2(\Pi,K) = \begin{cases} 0 & \text{if } \operatorname{char} K \neq 2, \\ 1 & \text{if } \operatorname{char} K = 2, \end{cases}$$

the Betti numbers $\beta_{3,(1,\dots,1)}$ and $\beta_{4,(1,\dots,1)}$ depend on K (use Corollary 1.5).

Remark 1.7. One can extend the previous results to an arbitrary affine semigroup ring $S \cong K[H]$ for which the invertible elements of H may form a non-zero group $H_0 \cong \mathbb{Z}^p$. Then $S(-h)$ and $S(-h-h_0)$ are isomorphic as multigraded modules for all $h \in H$ and $h_0 \in H_0$ so that the multigraded Betti numbers must be labelled by residue classes modulo H_0 if uniqueness of the shifts is desired. This fact is however compensated by the equality of Δ_h and Δ_{h+h_0}.

Let $\mathfrak{m}$ be the ideal generated by the non-invertible monomials. Then $\mathfrak{m}$ is a prime ideal and $S/\mathfrak{m} \cong K[H_0]$. The polynomial ring R must be replaced by $K[U_1^{\pm 1},\dots,U_p^{\pm 1}, X_1,\dots,X_n]$; it is mapped onto S by sending $U_1,\dots,U_p$ to a basis of the group of invertible monomials and $X_1,\dots,X_n$ to a minimal monomial system of generators of $\mathfrak{m}$. In equation (1) of Proposition 1.1, K is to be replaced by $K[U_1^{\pm 1},\dots,U_p^{\pm 1}]$, whereas equation (2) remains valid since the extension from K to $K[U_1^{\pm 1},\dots,U_p^{\pm 1}]$ is faithfully flat.

Remark 1.8. Let $\mathfrak{m}$ be the maximal ideal of T generated by the monomials $\neq 1$, and $\mathfrak{n}$ the maximal ideal of R generated by the indeterminates. Then $\mathscr{F}_{\mathfrak{m}}$ is a minimal free resolution of $T_{\mathfrak{m}}$ over $R_{\mathfrak{n}}$. Thus it follows from the Auslander–Buchsbaum formula and Theorem 1.3 that the inequality depth $T_{\mathfrak{m}} \geq i$ is valid or otherwise independently of K for $i = 0,1,2$ and, if $I = 0$, for $i = 3$. This also holds for the corresponding Serre property (S_i) which requires that depth $T_{\mathfrak{p}} \geq \min(i, \dim T_{\mathfrak{p}})$ for all prime ideals $\mathfrak{p} \in \operatorname{Spec} T$.

Let $\mathfrak{q}$ be the ideal generated by the monomials $\in \mathfrak{p}$. Then $\mathfrak{q}$ is a prime ideal, and

$$\dim T_{\mathfrak{p}} = \dim T_{\mathfrak{q}} + \dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}},$$

$$\operatorname{depth} T_{\mathfrak{p}} = \operatorname{depth} T_{\mathfrak{q}} + \dim R_{\mathfrak{p}}/\mathfrak{q}R_{\mathfrak{p}}.$$

(See [8, 1.2.2 and 1.2.4]. In [3, Section 1.5], we treat the case in which $\mathbb{Z}$ is the grading group; the general case can be proved by induction on the rank of the grading group.) Thus it suffices to consider $T_{\mathfrak{q}}$. This ring is of the form $(A/\mathfrak{a})_{\mathfrak{r}}$ where A is an affine semigroup ring (in the sense of Remark 1.7), $\mathfrak{a}$ is generated by monomials, and $\mathfrak{r}$ is the prime ideal generated by all non-invertible monomials. In view of Remark 1.7 we may therefore argue with depth again.

Remark 1.9. An analysis similar to that in Proposition 1.1 can be applied to the graded local cohomology of T. For a face F of the simplex Σ on $\{1,\dots,n\}$ let T_F denote the ring of fractions with respect to the multiplicative system generated by the elements y_i, $i \in F$. We define a complex $\mathscr{L}$ to be

$$\mathscr{L}\colon 0 \longrightarrow L^0 \xrightarrow{d_0} L^1 \xrightarrow{d_1} \dots \xrightarrow{d_{n-1}} L^n \to 0, \quad L^t = \bigoplus_{F\in\Sigma,\,|F|=t} T_F;$$

the differentiation d^t is given on the component $T_{F'} \to T_F$ by $\varepsilon(F,F')\cdot \mathrm{nat}$ if $F' \subset F$, and 0 otherwise.

Since $H^i_{\mathfrak{m}}(T) \cong H^i(\mathscr{L})$ (see [3, 3.5.6] for the local version of this isomorphism), the multigraded components of $H^i_{\mathfrak{m}}(T)$ can also be expressed by simplicial data. Given a degree $g \in \mathbb{Z}^m$, we set

$$\Omega_g = \{F\colon (S_F)_g = 0\} \quad \text{and} \quad \Theta_g = \Omega_g \cup \{F \in \Sigma\colon I_F \neq S_F\}.$$

Then Ω_g and Θ_g are simplicial complexes, and

$$H^i_{\mathfrak{m}}(T)_g = \tilde{H}^{i-1}(\Theta_g, \Omega_g, K).$$

It follows by similar arguments as above that the 'numerical structure' of $H^i_{\mathfrak{m}}(T)$ is independent from K for $i = 0, 1, n-1, n$; if $I = 0$, then $\Theta_g = \Sigma$, and one has independence for $i = 2$, too. Also Corollary 1.4 has an analogue for local cohomology. A further analysis of the case in which $R = S$ and I is generated by squarefree monomials yields Hochster's description of the local cohomology of Stanley–Reisner rings; see [10 or 3, 5.3.8].

Remark 1.10. Trung and Hoa [13] have shown that the triangulation of $\mathbb{RP}^2$ above can be 'realized' in the local cohomology of an affine semigroup ring. Their example shows that Theorem 1.3(c) cannot be extended to $i = n - 3$ and, simultaneously, that the assertion of the previous remark fails for $i = 3$.

2. The realization of simplicial complexes in semigroups

Let Π be a simplicial complex. In this section we will show that there exists a simplicial complex $\hat{\Pi}$ homotopically equivalent to Π that appears as a squarefree divisor complex Δ_h in a positive affine semigroup ring R. Furthermore we will exactly characterize those graphs (i.e. simplicial complexes of dimension at most 1) that can be realized as squarefree divisor complexes.

We first turn to the question of realizing a simplicial complex up to homotopy. Of course, this is just a matter of constructing a suitable semigroup; nevertheless, the language of commutative algebra is convenient in its presentation. In order to form $\hat{\Pi}$ we choose a new vertex v_F for each maximal face of Π, and let $\hat{\Pi}$ be the simplicial complex generated by the faces $F \cup \{v_F\}$ of the enlarged vertex set.

Let $V = \{1,\dots,n\}$ be the vertex set of Π. Then the ideal $I = I(\bar{\Pi}) \subset R = K[X_1,\dots,X_n]$ defining the Stanley–Reisner ring of $\bar{\Pi}$ is generated by the monomials $X^{V\setminus F}$ where F is extended over the maximal faces of Π. (As in Section 1, $\bar{\Pi}$ is the dual of Π.) We consider the Rees ring $\mathscr{R} = \mathscr{R}_I(R)$. It is the R-subalgebra of $R[T]$ generated by the elements $TX^{V\setminus F}$, F as above, and therefore is the semigroup ring generated over K by these monomials and the indeterminates X_i. Choose $h = (1,\dots,1) \in \mathbb{N}^{n+1}$; then the monomial with exponent vector h in $\mathscr{R}$ is $\mu = X_1 \dots X_n T$, and its decompositions into a product of irreducible elements are obviously given by

$$\mu = X_{i_1} \dots X_{i_k}(TX^{V\setminus F}), \quad F = \{i_1,\dots,i_k\} \text{ a maximal face of } \Pi.$$

Evidently $\hat{\Pi}$ can be identified with Δ_h if we let v_F correspond to $TX^{V\setminus F}$.

Let $\mathscr{S}$ be the normalization of $\mathscr{R}$. Then $\mathscr{S}$ is a normal affine semigroup ring whose underlying semigroup of monomials is the normalization of the semigroup generated by the elements X_i and $TX^{V\setminus F}$. We have

$$\mathscr{R} = \bigoplus_{i=0}^{\infty} I^i T^i \quad \text{and} \quad \mathscr{S} = \bigoplus_{i=0}^{\infty} J_i T^i,$$

where J_i is the integral closure of I^i. Since I is generated by squarefree monomials and thus an intersection of prime ideals, it is integrally closed. This implies that the decompositions of μ in $\mathscr{S}$ are exactly those in $\mathscr{R}$, and proves part (a) of the following theorem.

Theorem 2.1. *Let Π be a simplicial complex on the vertex set $\{1,\dots,n\}$. Then there exists a positive affine semigroup ring S and a monomial $\mu \in S$ such that the squarefree divisor complex of μ is homotopically equivalent to Π. Moreover, S can be chosen to be*

(a) *a normal subring of $K[X_1,\dots,X_n,T]$,*

(b) *a subring of $K[T]$, or*

(c) *a homogeneous subring of $K[T,U]$.*

Part (b) follows from the following proposition, which is a much more precise assertion than needed presently. For convenience we switch to additive notation.

Proposition 2.2. *Let $v, u_1, \dots, u_m \in \mathbb{N}^{n+1}$. Suppose that for $i = 1, \dots, k$ we have*

$$v = \sum_{j=1}^{m} b_{ij} u_j, \quad \text{with integers } b_{ij} \geq 0,$$

and that these are the only decompositions of v into sums of the u_j. Then there exist integers $d, d_1, \dots, d_m \in \mathbb{N}$ such that

$$d = \sum_{i=1}^{m} b_{ij} d_j, \quad \text{for } i = 1, \dots, k,$$

and such that no other such decomposition of d exists.

Proof. Let $g \in \mathbb{N}^{n+1}$, $g = (g_0, \dots, g_n)$. Then for a given $a \in \mathbb{N}$ we set $g(a) = \sum_{i=0}^{n} g_i a^i$. Note that $\mathbb{N}^{n+1} \to \mathbb{N}$, $g \mapsto g(a)$, is a homomorphism of semigroups. Thus for $i = 1, \dots, k$ it follows that

$$v(a) = \sum_{j=1}^{m} b_{ij} u_j(a).$$

This almost solves our problem. But we have to make sure that these are the only decompositions of $v(a)$ as sums of the $u_j(a)$.

To achieve this we choose a big enough. First we may assume that the last component of each u_j is larger than the other components of u_j. In fact, if this is not the case, then for all j we replace the u_j by $\tilde{u}_j = (u_{j0}, \dots, u_{jn}, \Sigma_k u_{jk}) \in \mathbb{N}^{n+2}$.

Now we choose $a \in \mathbb{N}$ with $a > v_n$. Suppose that $v(a) = \sum_{j=1}^{m} c_j u_j(a)$ with integers $c_j \geq 0$. Then

$$\sum_{i=0}^{n} v_i a^i = \sum_{i=0}^{n} \left(\sum_{j=1}^{m} c_j u_{ji} \right) a^i.$$

Assume $\sum_{j=1}^{m} c_j u_{nj} > v_n$. Then $\sum_{i=0}^{n} v_i a^i \geq (v_n + 1) a^n$, and so $\sum_{i=0}^{n-1} v_i a^i \geq a^n$. This is a contradiction since $v_i \leq v_n < a$ for all i. We conclude that $\sum_{j=1}^{m} c_j u_{jn} \leq v_n$. Therefore

$$\sum_{j=1}^{m} c_j u_{ij} \leq c_n < a$$

for $i, \dots, n$. Hence we see that both sums in the above equation represent the a-adic expansion of the same integer. This implies that $v_i = \sum_{j=1}^{m} c_j u_{ji}$ for $i = 1, \dots, n$.

In other words,

$$v = \sum_{j=1}^{m} c_j u_j,$$

as desired. □

In order to prove Theorem 2.1(c) we first replace Π by a pure simplicial complex Π' homotopically equivalent to Π. (A simplicial complex is pure if all its maximal faces have the same cardinality.) This can simply be done by adding new vertices to those maximal faces that are too 'small'. Thus we may assume that Π is pure. Then we construct $\mathscr{R}$ as above, and the previous proposition yields a 1-dimensional 'realization' $K[T^{u_1}, \dots, T^{u_n}]$ of Π. Since the equations resulting from the different decompositions of T^v are homogeneous, they carry over to the K-algebra $K[UT^{u_1}, \dots, UT^{u_n}]$.

Example 2.3. If we choose Π as the triangulation of $\mathbb{RP}^2$ as in Example 1.6, then the construction above yields a (normal) semigroup ring whose multigraded second Betti numbers are not independent of K. (Since $\Pi = \bar{\Pi}$, we can directly consider $I = I(\Pi)$.) The Rees algebra $\mathscr{R}$ is not normal. In fact, the element $X_1 \dots X_6 T^2$ is easily seen to be in the normalization of $\mathscr{R}$, but not in $\mathscr{R}$ itself. (One can show numerically that this element generates the normalization as an $\mathscr{R}$-algebra. The construction of $\tilde{\Pi}$ for this example was suggested to us by Günter Ziegler.)

The exact classification of the simplicial complexes Δ_h is presumably very difficult. We have however succeeded in describing all the graphs among them. Let Γ be a graph. Then we can pass to a homotopically equivalent graph Γ_0 by contracting the 'legs' of Γ into its 'body': we first remove the vertices of degree 1 and the edges adjacent to them, and iterate this procedure until we have obtained a graph Γ_0 in which all vertices have degree at least 2. We call Γ_0 the *body* of G. (The body of a tree is a single vertex.)

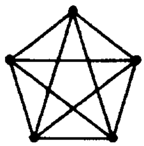
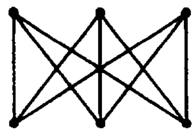
To simplify notation in what follows we will almost always identify a vertex of Δ_h with the irreducible element of H to which it corresponds.

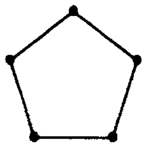
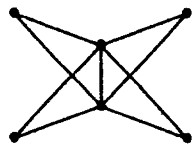
Theorem 2.4. *A graph Γ can be realized as the squarefree divisor complex Δ_h of an element h of an affine semigroup if and only if it satisfies the following conditions:*

(a) *each connected component of Γ_0 is one of the following graphs:*
  (i) *a complete graph $K(n)$, $n \geq 1$,*
  (ii) *a complete bipartite graph $K(m,n)$, $m,n \geq 1$,*
  (iii) *a cycle $Z(n)$, $n \geq 1$, or*
  (iv) *a graph of type $W(n)$ that is formed by joining the two vertices of the first component of a complete bipartite graph $K(2,n)$, $n \geq 1$;*

(b) *at most one of the connected components of Γ_0 is of type $K(n)$ with $n \geq 4$.*

We illustrate the types of graphs appearing in the theorem:

$K(5)$  $K(3,3)$ 

$Z(5)$  $W(4)$ 

We begin by showing that each of the graphs $K(n)$, $K(m,n)$, $Z(n)$, and $W(n)$ can be realized.

Proposition 2.5. *Let Γ be one of the graphs $K(n)$, $K(m,n)$, $Z(n)$, or $W(n)$. Then there exists a subsemigroup H of $\mathbb{N}$ and an element $h \in H$ such that $\Gamma \cong \Delta_h$.*

Moreover, if Γ is one of $K(1)$, $K(2)$, $K(3)$, $K(m,n)$, $Z(n)$, and $W(n)$, then H and h can be chosen such that h avoids any finite number of prime divisors.

Proof. For $K(n)$ we choose pairwise coprime numbers $q_1,\dots,q_n$ and set $u_i = \prod_{j\neq i} q_j$. The element $h = 2q_1\dots q_n$ has the decompositions $h = q_iu_i + q_ju_j$ so that its squarefree divisor complex indeed contains $K(n)$, and very elementary arguments of number theory show that these are the only decompositions of h in the semigroup generated by $u_1,\dots,u_n$.

For $K(m,n)$ let $p_1,\dots,p_m$ and $q_1,\dots,q_n$ be pairwise coprime natural numbers. We set $u_i = (\prod_{j\neq i} p_j, 0)$ and $v_k = (0, \prod_{l\neq k} q_l)$. Again it is easy to see that the element $h = (p_1\dots p_m, q_1\dots q_n)$ of the subsemigroup of $\mathbb{N}^2$ generated by the u_i and v_k has $K(m,n)$ as its squarefree divisor complex; in fact, its decompositions are given by $h = p_iu_i + q_jv_j$. According to Proposition 2.2 we find a 1-dimensional realization for the element $h' = p_1\dots p_m + aq_1\dots q_n + a^2(p_1\dots p_m + q_1\dots q_n)$ in the semigroup $H \subset \mathbb{N}$ generated by the elements

$$u_i' = \prod_{j\neq i} p_j + a^2 \prod_{j\neq i} p_j \quad \text{and} \quad v_k' = a\prod_{l\neq k} q_l + a^2 \prod_{l\neq k} q_l,$$

where a is a sufficiently large natural number. It is clear that by a suitable choice of a and $p_1,\dots,p_n$ we can avoid any given finite set of prime divisor for h'.

In order to realize $W(n)$ we pick pairwise coprime numbers $q_1, q_2, p_1,\dots,p_n$ and $r > q_1q_2$ coprime to each of $p_1,\dots,p_n$. The semigroup is generated by $u_1 = q_2p_1\dots p_n$, $u_2 = q_1p_1\dots p_n$, and $v_i = r\prod_{j\neq i} p_j$, $i = 1,\dots,n$. The element to be considered is $h = (q_1q_2 + r)p_1\dots p_n$; it has the decompositions $h = q_1u_1 + p_iv_i$ and $h = q_2u_2 + p_iv_i$. However, r belongs to the semigroup generated by q_1 and q_2 so that there is also a decomposition $h = r_1u_1 + r_2u_2$. Again we can avoid any finite number of prime divisors for h.

Let us show that h has no other decompositions than those already specified. In fact, in an equation $h = s_1u_1 + s_2u_2 + t_1v_1 + \cdots + t_nv_n$ each of the coefficients t_i must be divisible by p_i, as follows by taking residue classes modulo p_i. Therefore, and since $r > q_1q_2$, at most one of the t_i is non-zero, and necessarily $t_i = p_i$ in this case. If none of them is non-zero, then h is represented as a linear combination of u_1 and u_2. Otherwise, we get an equation $q_1q_2p_1 \dots p_n = s_1u_1 + s_2u_2$, and exactly one of the s_i is non-zero, $s_i = q_i$.

The case of $Z(n)$ is somewhat more complicated. We choose a number $\lambda \gg 0$, and set $c = 3\lambda^{n-1} - (-1)^n(\lambda - 1)$, and

$$u_i = (-1)^n(-\lambda)^{i-1} + c, \quad i = 1, \dots, n.$$

For $h = \lambda u_1 + u_2$ we have the representations

$$h = \lambda u_1 + u_2 = \lambda u_2 + u_3 = \cdots = \lambda u_{n-1} + u_n = 3u_n + (\lambda - 1)u_1$$

so that the squarefree divisor complex of h in the semigroup H generated by $u_1, \dots, u_n$ indeed contains Z_n. Furthermore, since $h \equiv (-1)^n \bmod \lambda$ we can avoid any finite number of prime divisors for h.

One easily sees that $u_i + u_j > u_k$ for all i, j, k so that $u_1, \dots, u_n$ form a minimal system of generators of H. Next note that $h = (\lambda + 1)c$. Hence $h = \sum a_iu_i$ implies $\sum a_i(-\lambda)^{i-1} \equiv 0 \bmod c$. If $a_i \le \lambda - 1$ for $i = 1, \dots, n-1$ and $a_n \le 2$, then it is not hard to see that $|\sum a_i(-\lambda)^{i-1}| < c$, whence we derive the contradiction $\sum a_i(-\lambda)^{i-1} = 0$. Therefore we have (i) $a_i \ge \lambda$ for some $i \le n-1$, or (ii) $a_n \ge 3$. Since $h - \lambda u_i = u_{i+1}$ we recover one of the representations above in case (i), and this is the only possibility since u_{i+1} is irreducible. In case (ii) we draw on the equation $h - 3u_n = (\lambda - 1)u_1$. Thus it remains to show that $(\lambda - 1)u_1$ has no other decomposition in H.

Assume that there exists a different decomposition in the case in which n is even. (The case of n odd is similar.) Then we have an equation

$$b(3\lambda^{n-1} - \lambda + 2) = 3\lambda^{n-1}\sum_{i=2}^{n} a_i + \sum_{i=2}^{n} a_i((-1)^{i-1}\lambda^{i-1} - \lambda + 1)$$

with $0 < b \le \lambda - 1$. Since $\lambda^{i-1} + \lambda + 1 < (3/2)\lambda^{n-1}$, we certainly have $\sum_{i=2}^{n} a_i < 2b$. On the other hand, taking residues modulo λ we get $2b \equiv \sum_{i=2}^{n} a_i \bmod \lambda$. The only remaining possibility is $2b = \lambda + \sum_{i=2}^{n} a_i$ so that $\sum_{i=2}^{n} a_i < b$. Note that $((-1)^{n-1}\lambda^{n-1} - \lambda + 1)a_i \le 0$ and that all the other terms $((-1)^{i-1}\lambda^{i-1} - \lambda + 1)a_i$ are less than $3\lambda^{n-2}$. Therefore $\sum_{i=2}^{n} a_i < b$ is impossible.

Finally we observe that $K(2) \cong K(1,1)$ and $K(3) \cong Z(3)$, and that for $K(1)$ we may pick H to be the semigroup generated by an arbitrary $q \in \mathbb{N}$, $q > 0$, and $h = mq$, $m \ge 1$. $\square$

The next lemma enables us to show that the conditions (a) and (b) of Theorem 2.4 are indeed sufficient for the realizability of a graph. To distinguish the vertices and the coefficients with which they appear in the representations of h we denote the coefficients by capital letters in the sequel.

Lemma 2.6. (a) *Suppose that Δ and Δ' are simplicial complexes with $\Delta = \Delta_h$ and $\Delta' = \Delta_{h'}$ for coprime elements h and h' of subsemigroups H and H' of $\mathbb{N}$. Then $hh' \in h'H + hH'$ has the disjoint union of Δ and Δ' as its squarefree divisor complex.*

(b) *Let the graph Γ be the squarefree divisor complex of $h \in H$, and let u be a vertex of Γ representing an irreducible element $\rho(u)$ of H. Suppose that $h = B\rho(u)$ with $B \geq 3$ or that $h = A\rho(u') + B\rho(u)$ with another vertex u' of Γ, $A \geq 1$ and $B \geq 2$. Then the graph Γ' that arises from Γ by the addition of a new vertex v and the edge $\{u, v\}$ can be realized as a squarefree divisor complex $\Delta_{h'}$. Moreover, $h' = C\rho'(u) + D\rho'(v)$ with $C \geq 1$, $D \geq 2$.*

Let now Γ be a graph satisfying the conditions of Theorem 2.4. Then, if Γ_0 contains a component of type $K(n)$ with $n \geq 4$, we start with its realization according to Proposition 2.5. Then we add all the other components of Γ_0, noting that the condition of Lemma 2.6 can always be satisfied. Now the body of Γ is complete, and we use Lemma 2.6(b) in order to attach its legs. (The condition of Lemma 2.6(b) is satisfied at every vertex of the realizations constructed in the proof of Proposition 2.5 and also satisfied at each 'new' vertex.)

Proof of Lemma 2.6. Part (a) becomes obvious when one takes congruences modulo h and h'.

For (b) we choose a prime number Q not dividing $(B-1)\rho(u)$ or $A\rho(u')+(B-1)\rho(u)$, respectively. Then we set $\rho'(v) = (B-1)\rho(u)$ or $\rho'(v) = A\rho(u') + (B-1)\rho(u)$, and $\rho'(w) = Q\rho(w)$ for all the vertices w of Γ. The semigroup H' is then generated by $\rho'(v)$ and the $\rho'(w)$. The element to be considered is $h' = Qh$.

In order to simplify the notation we write u for $\rho'(u), v$ for $\rho'(v)$, and more generally w for $\rho'(w)$. Note that $h' = u + Qv$ so that the last condition of (b) is satisfied. Suppose we have a representation

$$h' = \sum_{w \neq u} A_w w + A_u u + A_v v.$$

If $A_v = 0$, such a representation corresponds to a representation of h with the same coefficients. So suppose that $A_v > 0$. Since Q is prime to v, A_v is divisible by Q. Therefore $A_v = Q$; otherwise $A_v v > h'$. Thus $\sum_{w \neq u} A_w w + A_u u = u$, and in view of the hypothesis on $H \cong QH$ this is only possible with $A_u = 1$, $A_w = 0$ for $w \neq u$.

The proof of the irreducibility of the generators of H' uses similar arguments and can be left to the reader. $\square$

It remains to show the necessity of the conditions (a) and (b) of Theorem 2.4; the next lemma contains the crucial argument.

Lemma 2.7. *Suppose that $\Gamma = \Gamma_0$ is connected, and is not of type $Z(n)$ or $W(n)$. Let $\{a,b\} \neq \{b,c\}$, be edges of Γ, so that there are equations*

$$h = Aa + B_1 b = B_2 b + Cc, \quad A, B_1, B_2, C > 0.$$

Then $B_1 = B_2$, and therefore $Aa = Cc$.

We postpone the proof of Lemma 2.7.

For the following a piece of terminology will be useful. Let $\Gamma' \subset \Gamma$ and v a vertex of Γ'. Then we say that v *has constant coefficients in* Γ' if for all edges $\{v,w\} \in \Gamma'$ and a corresponding presentation $h = Vv + Ww$ with $V, W > 0$ the coefficient V is the same; Γ' *has constant coefficients* if all its vertices have constant coefficients.

For example, Lemma 2.7 says that the subgraph consisting of the vertices a, b, c and the edges $\{a,b\}$ and $\{b,c\}$ has constant coefficients. This holds though, in general, the coefficients A, B_1, B_2, C are not unique a priori. Suppose for example that we have another presentation $h = B_2' + C'c$; then $B_2' = B_1$ by the lemma, and hence $B_2' = B_2$. This implies $C' = C$. Thus Proposition 2.7 implies that the coefficients are unique.

We now show that the conditions (a) and (b) of Theorem 2.4 are necessary. So, let C be a connected component of Γ_0. Then C is the restriction of Γ to $\operatorname{vert}(C)$. Restricting the semigroup to the subsemigroup generated by the irreducible elements represented by $\operatorname{vert}(C)$ we may assume in the following that $C = \Gamma$, in other words, $\Gamma = \Gamma_0$ and is connected. (There is nothing to show if C is a tree.) Suppose that Γ is not of type $Z(n)$ or $W(n)$. Then we must show Γ is of type $K(n)$ or $K(m,n)$. Define the equivalence relation $\sim$ on $\operatorname{vert}(\Gamma)$ by setting $v \sim w$ if there is a path $v = v_0, \ldots, v_n = w$ in Γ with n even. Obviously $\operatorname{vert}(\Gamma)$ decomposes into at most 2 classes modulo $\sim$.

It follows from Lemma 2.7, that Γ has constant coefficients. Therefore, if $v_1, \ldots, v_u$ are the vertices of Γ with coefficients $V_1, \ldots, V_u$, then $V_i v_i = V_j v_j$ whenever $v_i \sim v_j$. Hence we either have a single chain of equations

$$V_1 v_1 = V_2 v_2 = \cdots = V_u v_u$$

or two such chains

$$V_{i_1} v_{i_1} = \cdots = V_{i_s} v_{i_s} \quad \text{and} \quad V_{j_1} v_{j_1} = \cdots = V_{j_t} v_{i_t}, \quad V_{i_1} v_{i_1} \neq V_{j_1} v_{j_1}.$$

In the first case $\Gamma \cong K(w)$, and in the second $\Gamma \cong K(s,t)$ as was to be shown. The element $h \in H$ with $\Gamma = \Delta_h$ is $2V_1 v_1 = V_i v_i + V_j v_j$ in the first and $V_{i_1} v_{i_1} + V_{j_1} v_{j_1}$ in the second case.

The last observation implies the necessity of Theorem 2.4(b). In fact, if $\Gamma = \Delta_h$ contains two subgraphs Γ' and Γ'' of type $K(m)$ and $K(n)$ respectively, $m, n \geq 4$, then $h = 2Vv = 2Ww$ where $v \in \operatorname{vert}(\Gamma')$, $w \in \operatorname{vert}(\Gamma'')$, and V and W are the coefficients of v and w. It follows that $h = Vv + Ww$ so that the edge $\{v, w\}$ belongs to Γ. This concludes the proof of Theorem 2.4.

The next lemma contains the elementary observation that is the key to the classification of graphs realizable as squarefree divisor complexes.

Lemma 2.8. *Suppose that $\Gamma = \Delta_h$ is a graph containing the edges $\{a,b\}$, $\{b,c\}$, $\{c,d\}$ where a,b,c are pairwise different, but $a = d$ is not excluded, and*

$$h = Aa + B_1 b = B_2 b + C_2 c = C_3 c + Dd \quad \text{with } B_1 > B_2.$$

If $a \neq d$, then $C_2 > C_3$, and if $a = d$, then $C_2 \geq C_3$.

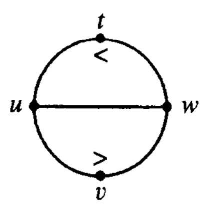
Proof. We have $C_2 c = Aa + (B_1 - B_2)b$. If $C_2 \leq C_3$, then $h = Aa + (B_1 - B_2)b + (C_3 - C_2)c + Dd$. Now, if $a \neq d$, then $\{a,b,d\} \in \Delta_h$, in contradiction to the fact that Δ_h is a graph, and if $a = d$ we get $\{a,b,c\} \in \Delta_h$, unless $C_2 = C_3$. □

Let $v_1, \dots, v_n$ be the vertices of a subgraph $C \cong Z(n)$ of Γ. Then we say that C is an n-cycle in G, and a path P joining two different vertices v_i and v_j is called a *diagonal* of C if P and C have no common edge.

Lemma 2.9. *Let $\Gamma = \Delta_h$ be a graph, and C an n-cycle in G with a diagonal D.* (a) *If $n \geq 4$, then C has constant coefficients, and* (b) *if $n \geq 5$, then $C \cup D$ has constant coefficients.*

Proof. Let u and w be the vertices of C joined by the diagonal. If u and w are neighbours on C, then we exchange the role of the edge $\{u,w\}$ and the diagonal. Since in this case D contains a vertex different from u and w, we may assume that there is a vertex different from u and w on each of the arcs of C joining u and w. (Since the length of the cycle increases under this operation, the hypothesis on n has even been improved.)

Assume C does not have constant coefficients. Then the inequality at any vertex propagates around the cycle according to Lemma 2.8:



Suppose first that $n \geq 5$. Then we may assume that there is a further vertex between v and w. For a walk from v to t we then have two choices: either counterclockwise along the cycle, or via the diagonal to u and then clockwise along the cycle. The second choice in conjunction with Lemma 2.8 yields the clockwise inequality $\geq$ at t, and thus a contradiction.

If $n = 4$, then we similarly obtain a contradiction if the diagonal contains a vertex other than u, w. Thus there remains only the case in which the diagram above shows exactly all the vertices and edges of $C \cup D$. For the following discussion and the proofs of the next lemmas the following symbols will be handy. Let $\{a,b\}$ and $\{b,c\}$ be edges of Γ. Then we write

$$a(b,>)c \quad \text{if } h = Aa + B_1 b = B_2 b + Cc \ \text{ implies } B_1 > B_2, \text{ and}$$
$$[a(b,>)c] \quad \text{if } h = Aa + B_1 b = B_2 b + Cc \ \text{ with } B_1 > B_2 \text{ is possible.}$$

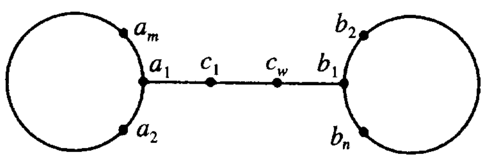
(The distinction between $a(b,>)c$ and $[a(b,>)c]$ is necessary since the coefficients are not necessarily unique a priori; furthermore, in each situation, $a(b,>)c$ must be understood relative to the choices made beforehand.)

With the notation just introduced, our assumption is $[w(t,>)u]$. According to Lemma 2.8 we have $t(u,\geq),w$ and $v(w,\geq)u$, and thus $>$ is excluded for each of them. Hence $t(u,=),w$ and $v(w,=)u$, and therefore $\{t,v\} \in \Gamma$. Observe that $w(u,>),v$, whence by Lemma 2.8 we consecutively obtain $u(v,>)t$, $v(t,>)w$, and $t(w,>)u$. On the other hand, $v(w,=)u$ and $v(w,>)t$, which altogether is an impossible constellation.

For (b) we note that each edge of $C \cup D$ now lies on an m-cycle, $m \geq 4$, with a diagonal, and we apply part (a) to each of these cycles. (Note that the latter does not apply to a 4-cycle with a diagonal consisting of a single edge as demonstrated by the graphs $W(n)$.) □

We need a similar statement in the situation where two cycles are joined by a straight line.

Lemma 2.10. *Suppose that $\Gamma = \Delta_h$ contains a subgraph Σ according to the following figure (in the following such a graph is called a* double loop*):*



(We require that $m,n \geq 3$, but allow the cases $w=0$ or even $a_1 = b_1$.) Then Σ has constant coefficients.

Proof. Suppose that the assertion does not hold. Then it follows from Lemma 2.8 that it does not even hold in one of the cycles C_i, say, C_1. Now, if $m \geq 3$, then the inequality propagates over C_1, and also into the cycle C_2, clockwise as well as counterclockwise. Regardless of whether $n = 3$ or $n > 3$, this yields a contradiction.

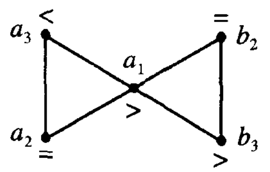
Thus we are left with the case $m = 3$. If we assume that $[a_3(a_2,>)a_1]$ or $[a_2(a_3,>)a_1]$, then we obtain the same contradiction as before. Thus, and by symmetry, we may assume that $[a_1(a_3,>)a_2]$ and $a_1(a_2,\geq)a_3$. Then $a_1(a_2,=)a_3$ by Lemma 2.8.

The two cases in which $a_1 \neq b_1$ and $a_1 = b_1$ respectively are slightly different. We treat the second, leaving the first to the reader.

Note that $[b_n(a_1,>)a_2]$ is impossible since it would contradict $a_1(a_2,=)a_3$. If $[b_n(a_1,=)a_2]$, then Γ also contains the edge $\{b_n, a_2\}$. In that case the edges $\{a_1, a_3\}$

and $\{a_3, a_2\}$ lie on a 4-cycle with the diagonal $\{a_2, a_1\}$ so that $a_1(a_3,=),a_2$ by Lemma 2.9, and this contradicts our initial assumption.

Thus it remains to discuss the case $a_2(a_1,>)b_n$. Then, if $n \geq 4$, this inequality travels along C_2 and back into C_1 where we obtain contradictory inequalities. So we are left with the following situation (note that $[b_3(b_2,>)a_1]$ is incompatible with $a_1(a_2,=),a_3$):



The equalities at a_2 and b_2 force the equality $a_2(a_1,=)b_2$ so that $\{a_3,b_2\},\{a_2,b_3\} \in \Gamma$. Lemma 2.9 then yields the final contradiction. □

Now Lemma 2.7 follows from the Lemmata 2.9 and 2.10 and the next, purely graph theoretic argument.

Proposition 2.11. *Suppose that $\Gamma = \Gamma_0$ is a connected graph that is neither of type $Z(n)$ nor $W(n)$. Suppose that $\{a,b\} \neq \{b,c\}$ are edges of Γ. Then $\{a,b\}$ and $\{b,c\}$ are contained in a subgraph that is*

(i) *an n-cycle, $n \geq 4$ with a diagonal, or*

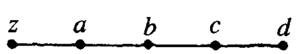
(ii) *the union of an n-cycle, $n \geq 5$, and a diagonal thereof, or*

(iii) *a double loop.*

Proof. For a vertex v of Γ we denote by $N(v)$ the set of neighbours of v.

Our first observation is that every cycle C contained in Γ has a diagonal or is one of the cycles of a double loop. In fact, since Γ is not a cycle, there exists a vertex w outside C, and we choose a shortest path connecting w with one of the vertices v of C. Then we walk from v to w along the path choosen, and continue our walk without ever turning back at any vertex (such a walk is *interesting*). This is possible since $\Gamma = \Gamma_0$ contains no blind alley. If our walk reaches C before it intersects itself, then C has a diagonal. Otherwise we have found a double loop with C as one of its cycles. We now distinguish several cases.

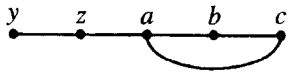
(i) There exists vertices $z \in N(a)$, $z \neq b,c$, and $d \in N(c)$, $d \neq a,b$ (but possibly $z = d$):



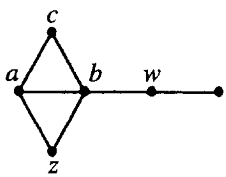
We then start an interesting walk from c via b and a to z and beyond, and simultaneously an interesting walk from a to d and beyond. In whatever way these two walks intersect each other or themselves, we always obtain the desired conclusion. (If we have formed a single cycle, then we use the previous observation; note that this cycle has at least 4 vertices.)

By symmetry we may from now on assume that $N(c)=\{a,b\}$.

(ii) There exists $z\in N(a)$, $z\neq b,c$, and there exists $y\in N(z)$, $y\neq a,b$. By a similar argument as in case (i) we obtain that $\{a,b\}$ and $\{b,c\}$ are contained in a double loop or in the union of an n-cycle, $n\geq 5$, with a diagonal. (Note that $y\neq c$.)



(iii) There exists $z\in N(a)$, $z\neq b,c$, but $N(z)=\{a,b\}$ for all $z\in N(a)$. If further $N(w)=\{a,b\}$ for all $w\in N(b)$, then $\Gamma\cong W(m)$ for some m, a case we have excluded by hypothesis on Γ. Thus there exists $w\in N(b)$, $N(w)\neq\{a,b\}$.



We then arrive at the same conclusion as in case (ii).

(iv) $N(a)=\{b,c\}$. Since $\Gamma\not\cong Z(3), b$ has a further neighbour $w\neq a,c$, and $\{a,b\}$ and $\{b,c\}$ are contained in a double loop. □

We now illustrate the difference between the divisibility theory of an arbitrary affine semigroup and that of a normal one by describing the graphs that can be realized in normal semigroups.

Theorem 2.12. *Suppose that H is a normal affine semigroup, and that $\Gamma=\Delta_h$ is a graph for some $h\in H$. Let $n+1$ be the number of connected components of Γ. Then n of these are isomorphic to •—•, and the last is one of •, •—•, or •—•—•. Conversely, each such graph can be realized in a normal semigroup.*

We first show that all the graphs listed can indeed be realized. Since we want to argue ring-theoretically we use multiplicative notation. In each of the following cases consider the semigroup generated by the elements given, subject to the relations

(i) $\quad x_1y_1=\cdots=x_ny_n=z^2,$

(ii) $\quad x_1y_1=\cdots=x_ny_n=x_{n+1}y_{n+1},$

(iii) $\quad x_1y_1=\cdots=x_ny_n=uv^2=vw^2.$

It is not hard to see that these relations in each case define a complete intersection R. Furthermore one checks that Serre's condition (R_1) is satisfied. Observe R is graded (choose $\deg x_i=\deg y_i=\deg z=3$ and $\deg u=\deg v=\deg w=2$). Altogether it follows that R is a normal domain. Since it is defined by binomial equations, it is a semigroup ring; see [6]. In each case h is the element represented by a single term in the corresponding equation.

For the proof of the necessity the following lemma is crucial. We again switch to additive notation.

Lemma 2.13. *Let H be a normal semigroup, $h \in H$, and suppose that $\Gamma = \Delta_h$ is a graph. Let Γ contain the edges $\{a,b\} \neq \{b,c\}$, so that there are equations*

$$h = Aa + B_1 b = B_2 b + Cc, \quad A, B_1, B_2, C > 0.$$

Then $B_2 = B_1 + 1$ and $C = 1$, or $B_1 = B_2 + 1$ and $A = 1$.

Proof. Normality of H implies $\operatorname{rank}(\mathbb{N}a + \mathbb{N}c) = 2$, so that $B_1 = B_2$ is impossible. We assume that $B_1 < B_2$ and must show that the first alternative applies. Set $B = B_2 - B_1$. Then $Aa = Bb + Cc$.

Let us first assume that $B \geq A$. Then we have $A(a-b) = (B-A)b + Cc \in H$. Since H is normal, $a - b \in H$, and the irreducibility of a implies the contradiction $a = b$.

It follows that $A > B$, and similarly we have $A > C$; set $D = \max(B, C)$. Then

$$(A - D)a + (D - B)b + (D - c)c = D(b + c - a) \in H$$

so that $b + c \in a + H$. Let $g = b + c - a$.

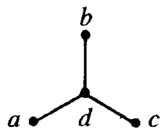
If $B \geq 2$ and $C \geq 2$, then $h = (B-1)b + (C-1)c + a + g$, whence $\{a,b,c\} \in \Delta_h$, and that is excluded by our hypothesis.

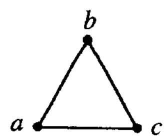
Next assume $B \geq 2$ and $C = 1$, that is, $Aa = Bb + c$. Since $h = (B-1)b + a + g$, only the irreducible elements a and b can appear in a decomposition of g: $g = Xa + Yb$. If $Y > 0$, then the impossible equation $c = (X+1)a + (Y-1)b$ follows. Thus $Y = 0$, $g = Xa$, and

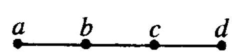
$$b + c = (X+1)a, \qquad Bb + c = Aa.$$

Since $\operatorname{rank} \mathbb{N}a + \mathbb{N}b + \mathbb{N}c = 2$, we obtain $B = 1$, and thus a contradiction. Similarly one sees that $B = 1$, $C \geq 2$ is impossible. $\square$

Lemma 2.14. *Let Γ be as in Lemma 2.13. Then Γ does not contain one of the following subgraphs:*

$V(3)$:


$Z(3)$:


$L(3)$:


Proof. Suppose Γ contains $V(3)$. Then $h = Aa + D_1 d = Bb + D_2 d = Cc + D_3 d$ where D_1, D_2, D_3 are pairwise distinct. We may assume $D_1 < D_2 < D_3$. According to Lemma 2.13 we have $D_1 = D_2 - 1$, $D_1 = D_3 - 1$, and $D_2 = D_3 - 1$, and this is obviously impossible.

Assume that Γ contains $Z(3)$. By symmetry, and in view of Lemma 2.13, we then have equations $h=2a+b=2b+c=2c+a$. Then $\operatorname{rank}(\mathbb{N}a+\mathbb{N}b+\mathbb{N}c)=1$, and again we have a contradiction.

If Γ contains $L(3)$, then one has equations $h=Aa+B_1b=B_2b+C_2c=C_3c+Dd$. If $B_1>B_2$, then $C_2>C_3$ in view of Lemma 2.8. By symmetry we may therefore assume $B_1<B_2$. Then $B_2=B_1+1$, $C_2=1$, $C_3=C_2+1=2$, and $D=1$. (This follows from Lemma 2.13.) Thus

$$h=Aa+B_1b=(B_1+1)b+c=2c+d.$$

This yields

$$(B_1+1)Aa=(B_1+1)b+(B_1+1)c=(B_1+2)c+d.$$

A similar discussion as in the proof of Lemma 2.13 implies that $c+d\in a+H$. Then $h=c+a+g$ with $g\in H$, and Γ contains the edge $\{a,c\}$. Since Γ does not contain $Z(3)$, as just seen, we have derived a final contradiction. $\square$

It is now clear that Δ_h can only have connected components as described in Theorem 2.12. We leave it to the leader to show that at most one component can be of a type different from $\bullet\!\!-\!\!\!-\!\!\bullet$. (The argument is similar as that used in the proof of Lemma 2.13.)

3. The vanishing of homology of squarefree divisor complexes

In this concluding section we show that the reduced simplicial homology of a squarefree divisor complex Δ_h of a normal semigroup vanishes up to an index which can be expressed in terms of h and the semigroup. We apply this result to some specific examples which arise from Segre product constructions. Of particular interest will be the so-called chessboard complexes which occur in this context. We are grateful to Günter Ziegler for providing us with information about the literature on chessboard complexes.

Let H be a semigroup, $S=K[H]$ the semigroup ring, $\mathscr{F}$ the minimal multigraded free resolution of S with respect to the minimal representation $S=R/I$. As observed in Section 1, $\mathscr{F}$ is multigraded: its ith free module F_i decomposes into a direct sum $\bigoplus_{h\in H}R(-h)^{\beta_{ih}}$ with $\beta_{ih}=\dim_K\tilde{H}_{i-1}(\Delta_h)$. Here and in the following we always take coefficients in K, unless indicated otherwise.

We now assume that H is normal, and denote by $\operatorname{relint}H$ the relative interior of H. We will quote several results about $\mathbb{Z}$-graded canonical modules from [3]. These statements can be carried over accordingly to the multigraded case.

By a theorem of Hochster (see [3, Theorem 6.3.5(a)]), the semigroup ring S is Cohen–Macaulay, and by a theorem of Danilov and Stanley (see [3, Theorem 6.3.5(b)]), the ideal J is generated by the elements Y^h, $h\in\operatorname{relint}H$, is the multigraded canonical module ω_S of S. On the other hand (see [3, Proposition 3.6.12]), the canonical module can also be computed in terms of $\mathscr{F}$. Indeed, suppose H is generated by the irreducible elements $h_1,\dots,h_n$, and that $d=\dim S$. Set $p=n-d$; then F_p is last non-vanishing free module in the resolution, and ω_S is the cokernel of $F_{p-1}^\vee \to F_p^\vee$ (where $N^\vee$ denotes the multigraded ω_R-dual of the multigraded R-module N). Hence if

$$F_p = \bigoplus_{i=1}^{t} R(-h_{pi}),$$

then, since $\omega_R \cong R(-\sum_{j=1}^n h_j)$, J is generated by the elements

$$Y^{\sum_{j=1}^n h_j - h_{pi}}, \quad i=1,\dots,t.$$

In the next proposition we write H additively, and introduce some more notation: we denote by $H_{>0}$ the set of all non-zero elements of H, and for any two subsets A and B of H we set $A+B=\{a+b\colon\ a\in A,\, b\in B\}$. Furthermore, we write iA for the set $A+\cdots+A$, where we add i copies of A.

Proposition 3.1. *With the notation and hypotheses introduced we have*

$$\tilde{H}_{i-1}(\Delta_h)=0 \quad \textit{if} \ \sum_{j=1}^n h_j \notin h+\operatorname{relint} H+(p-i)H_{>0}.$$

In particular, if $q\le p$ is the largest integer such that $\sum_{j=1}^n h_j \notin h+\operatorname{relint} H+(p-q)H_{>0}$, then $\tilde{H}_{i-1}(\Delta_h)=$ for all $i\le q$.

Proof. Suppose $\beta_{ih}\neq 0$. By our assumption S is Cohen–Macaulay, so that the defining ideal of S is perfect. In this situation one has: if $\beta_{ih}\neq 0$, then there exists $g_1\in h+H_{>0}$ such that $\beta_{i+1g_1}\neq 0$. In fact otherwise, the matrix φ defining the differential $F_{i+1}\to F_i$ would have a zero row. Thus in the R-dual of $\mathscr{F}$ there would appear a matrix with a zero column, which is a contradiction, since $\mathscr{F}^\vee$ is a minimal free resolution of ω_S.

By induction on the length of the resolution we now see that there exists an integer s, $1\le s\le t$, such that $h_{ps}\in h+(p-i)H_{>0}$. Thus, since $\sum_{j=1}^n h_j - h_{ps}\in \operatorname{relint} H$, we see that $\sum_{j=1}^n h_j \in h+\operatorname{relint} H+(p-i)H_{>0}$, a contradiction. □

If S is Gorenstein, then ω_S is a cyclic module, and hence $\operatorname{relint} H = g+H$. Therefore we obtain

Corollary 3.2. *Suppose $S=K[H]$ is Gorenstein, and* $\operatorname{relint} H=g+H$. *Then for all $i<p$ one has,*

$$\tilde{H}_{i-1}(\Delta_h)=0 \quad \textit{if} \ \sum_{j=1}^n h_j - g \notin h+(p-i+1)H_{>0},$$

and moreover

$$\tilde{H}_{p-1}(\Delta_h) = 0 \quad \text{if } h \neq \sum_{j=1}^{n} h_j - g.$$

In particular, Δ_h *is acyclic, if* $\sum_{j=1}^{n} h_j - g \notin h + H$.

Proof. We only need to explain why in the above statement we can write $(p-i+1)H_{>0}$ instead of $(p-i)H_{>0}$ as in Proposition 3.1. The reason is that the resolution of a Gorenstein complex is self-dual, which in turn implies that the last shift in the resolution differs from the shifts in F_{p-1} by elements in $2H_{>0}$. □

In the proof of Corollary 3.2 we used that the resolution of a Gorenstein ring is self-dual. This implies in particular that F_p is cyclic with shift, $f = \sum_{j=1}^{n} h_j - g$, and that $\beta_{ih} = \beta_{p-i,f-h}$ for all i and h. Hence

Corollary 3.3. *Suppose* $S = K[H]$ *is Gorenstein, and* $\operatorname{relint} H = g + H$. *Set* $f = \sum_{j=1}^{n} h_j - g$; *then*

$$\tilde{H}_{i-1}(\Delta_h) = \tilde{H}_{p-i-1}(\Delta_{f-h})$$

for all i and h.

We now want to apply Proposition 3.1 and Corollary 3.2 to some specific examples. To begin with we note that certain chessboard complexes may be realized as squarefree divisor complexes. Recall that the collection of all admissible rook configurations on a general $n \times n$ chessboard is called a chessboard complex, and is denoted by $\Delta_{n,n}$. An admissible rook configuration is any non-taking placement of rooks.

For more information about the history and significance of chessboard complexes in combinatorics we refer the reader to [2].

For the realization of these complexes we fix a field K, and consider the semigroup $H_{n,n}$ generated by all monomials $y_{ij} = Y_i Z_j$ in $K[Y_1, \ldots, Y_n, Z_1, \ldots, Z_n]$. Let $y = \prod_{i=1}^{n} Y_i \prod_{j=1}^{n} Z_j$; then $y \in H_{n,n}$ and $\Delta_h = \Delta_{n,n}$.

This example can be extended in many directions, observing that $K[H_{n,n}]$ is the Segre product $A * B$ of two polynomial rings A and B. More generally, we will consider the Segre product of two normal homogeneous semigroup rings. We call a semigroup $H \subset \mathbb{N}^m$ *homogeneous* if there is a constant $c = c(H)$ such that $|h| = \sum_i h_i = c$ for all irreducible elements $h = (h_1, \ldots, h_m)$ of H. If this is the case, then the absolute value $|h|$ of each element of H is a multiple of c, and $K[H]$ has the structure of a homogeneous K-algebra: $K[H] = \bigoplus_{i \geq 0} (\bigoplus_{|h|=ic} Kh)$. Thus for $y = Y^h \in K[H]$ one sets $\deg y = |h|/c$.

Applying the results [7, Theorem 4.2.3] of Goto and Watanabe to homogeneous semigroup rings, we obtain

Proposition 3.4. *Let $H_1 \subset \mathbb{N}^m$ and $H_2 \subset \mathbb{N}^n$ be two normal homogeneous semi-groups with constants $c_1 = c(H_1)$ and $c_2 = c(H_2)$. Let $H = H_1 * H_2 \subset \mathbb{N}^{m+n}$ be the semigroup generated by the elements*

$$\{(h_1, h_2)\colon\ h_i \in H_i,\ |h_i| = c_i,\ i = 1,2\}.$$

Then

(a) *H is a normal homogeneous semigroup, and there is an isomorphism of graded k-algebras*

$$K[H_1] * K[H_2] \cong K[H].$$

(b) $\operatorname{relint} H = \{(h_1, h_2)\colon\ h_i \in \operatorname{relint} H_i,\ |h_i| = jc_i,\ i = 1,2,\ j = 1,2,\ldots\}$.

It is often more convenient to express Proposition 3.4 in terms of the associated rings and modules. Say $K[H_1] \subset K[Y_1, \ldots, Y_m]$ is generated by the monomials $y_1, \ldots, y_{r_1}$ of degree c_1, and $K[H_2] \subset K[Z_1, \ldots, Z_n]$ by the monomials $z_1, \ldots, z_{r_2}$ of degree c_2. Then $K[H_1] * K[H_2] \subset K[Y_1, \ldots, Y_m, Z_1, \ldots, Z_m]$ is generated by the monomials $y_i z_j$, $i = 1, \ldots, r_1$, $j = 1, \ldots, r_2$. Moreover Proposition 3.4(b) says that $\omega_{K[H]}$ is generated by the set of monomials $\mathscr{M} = \{yz\colon\ y \in \omega_{K[H_1]},\ z \in \omega_{K[H_2]},\ \deg y = \deg z\}$. A minimal set of generators of $K[H]$ is given by all monomials $yz \in \mathscr{M}$ for which either y is a minimal generator of $\omega_{K[H_1]}$, or z is a minimal generator of $\omega_{K[H_2]}$.

Remark 3.5. *Chessboard complexes with multiplicities.* We let $H_{m,n} = \mathbb{N}^m * \mathbb{N}^n$, and choose $h \in \mathbb{N}^{m+n}$, $h = (a_1, \ldots, a_m, b_1, \ldots, b_n)$ with $\sum_i^m a_i = \sum_i^n b_i$. Then $h \in H_{m,n}$, and Δ_h may be identified with rook configurations on an $m \times n$-chessboard where for $i = 1, \ldots, m$ and $j = 1, \ldots, n$ it is allowed to place at most a_i rooks on the ith row and b_j rooks on the jth column of the chessboard.

Let us compute $\operatorname{relint} H_{m,n}$: obviously one has $\operatorname{relint} \mathbb{N}^r = (1, \ldots, 1) + \mathbb{N}^r$ for any $r \geq 1$. Thus, if we assume that $m \leq n$, then $\omega_{K[H_{m,n}]}$ is generated by the monomials $Y^c Z_1, \ldots, Z_n$ with $|c| = n$, and all components of c positive. In other words, $\operatorname{relint} H_{m,n}$ is generated by the elements $(c_1, \ldots, c_m, 1, \ldots, 1) \in \mathbb{N}^{m+n}$ with $c_i > 0$ and $\sum_{i=1}^m c_i = n$. Thus Proposition 3.1 implies that Δ_h is acyclic if $a_i > n$ for some i, $b_j \geq m$ for some j, or $\sum_{i=1}^m a_i > mn - n$.

Remark 3.6. *Higher dimensional chessboard complexes.* We let $H_{n_1,\ldots,n_r} = \mathbb{N}^{n_1} * \cdots * \mathbb{N}^{n_r}$ with $n_1 \leq \cdots \leq n_r$, and choose $h = (a_{11}, \ldots, a_{1n_1}, \ldots, \ldots, a_{r1}, \ldots, a_{rn_r})$ in $\mathbb{N}^{n_1 + \cdots + n_r}$, where the sums $\sum_{i=1}^{n_j} a_{ji}$ are independent of j. Then $h \in H_{n_1,\ldots,n_r}$, and Δ_h may be identified with the chessboard complex which is the collection of rook placements on the r-dimensional chessboard of shape $n_1 \times n_2 \times \cdots \times n_r$ where at most a_{ij} rooks belong to the $(r-1)$-dimensional hyperplane orthogonal to the ith axis of the chessboard, and intersecting this axis in a distance of j units from the origin.

We leave it to the reader to formulate a general condition for the acyclicity of Δ_h, and consider here only the special case that $n_1 = \cdots = n_r = n$. We have $\operatorname{relint} H_{n,\ldots,n} = g + H_{n,\ldots,n}$ with $g = (1, \ldots, 1)$. Since $K[H_{n,\ldots,n}]$ is an iterated Segre product, we may compute its dimension (assuming $n \geq 2$) according to [7, Proposition 4.2.4], and obtain $\dim K[H_{n,\dots,n}] = rn - (r-1)$. Hence referring to the notation of Corollary 3.2 we get $p = n^r - rn + (r-1)$, and it follows that $\tilde{H}_{i-1}(\Delta_h) = 0$ for all i such that

$$(n^{r-1}-1, \dots, n^{r-1}-1) \notin h + (n^r - rn + (r-1) - i + 1)H_{>0},$$

where $H = H_{n,\dots,n}$. This condition is satisfied if the absolute value $rn(n^{r-1}-1)$ of $(n^{r-1}-1, \dots, n^{r-1}-1)$ is less than the absolute value of any element of $h + (n^r - rn + (r-1) - i + 1)H_{>0}$. One easily checks that this is the case if

$$i \leq \frac{1}{r} \sum_{s,t} a_{st} - (r-1)(n-1).$$

For example, if $r = 2$ and $a_{st} = a$ for all r and s, then $\tilde{H}_{i-1}(\Delta_h) = 0$ for all $i \leq (a-1)n + 1$.

Some non-vanishing simplicial homology groups of chessboard complexes have been computed. It has been shown [2, Proposition 2.3] that $\tilde{H}_2(\Delta_{5,5}, \mathbb{Z}) \cong \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_3$. This implies that $\beta_3(K[H_{5,5}])$ depends on the characteristic. Note that $K[H_{5,5}]$ is the determinantal ring defined by the 2-minors of a 5×5 matrix. Thus this approach yields a different proof of a well-known result of Hashimoto [9] who first showed that the resolution of determinantal rings may depend on the characteristic of the base field. Anderson [1] showed a similar result for symmetric matrices directly by a machine computation of the homology of a squarefree divisor complex.

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